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Path integration in conical space

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ABSTRACT

Quantum mechanics in conical space is studied by the path integral method. It is shown that the curvature effect gives rise to an effective potential in the radial path integral. It is further shown that the radial path integral in conical space can be reduced to a form identical with that in flat space when the discrete angular momentum of each partial wave is replaced by a specific non-integral angular momentum. The effective potential is found proportional to the squared mean curvature of the conical surface embedded in Euclidean space. The path integral calculation is compatible with the Schrödinger equation modified with the Gaussian and the mean curvature.

1. Introduction

In recent years there has been considerable interest in quantum mechanics in the field of topological defects (see, e.g., [1]). Although the notion of defects in physics was originally associated with crystalline irregularities, it has been extended to more general topological structures such as entangled polymers, liquid crystals, magnetic vortices, anyons, monopoles, cosmic strings, domain walls, and so on. Standard approaches to particle-defect interactions in quantum mechanics are to solve appropriate Schrödinger equations with relevant boundary conditions. The space surrounding a defect is often characterized by torsion and curvature. The torsion may be globally treated through boundary conditions in connection with the topological nature of the defect as in the case of the Aharonov–Bohm effect, whereas the information of curvature would have to be fully contained in the Schrödinger equation. In curved space, the Schrödinger equation is usually expressed in the form,

$$\left\{-\frac{\hbar^2}{2M}\Delta + V_c(\mathbf{r}) + V(\mathbf{r})\right\}\psi(\mathbf{r}) = i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}) \tag{1}$$

where Δ is the Laplace–Beltrami operator, $V_c(\mathbf{r})$ is the so-called curvature term and $V(\mathbf{r})$ is any external potential. Historically, Podolsky [2] defined the Schrödinger equation in curved space without the curvature term, namely $V_c(\mathbf{r}) = 0$. Comparing with the path integral formulation, DeWitt [3] asserted that the curvature term is proportional to the Ricci scalar curvature, that is, $V_c(\mathbf{r}) = g\hbar^2 R(\mathbf{r})$ where g is a constant. Moreover, Jensen and Koppe [4], and da Costa [5] argued, if the particle in question is constrained to move on a two-dimensional curved surface, the curvature term must be give by

$$V_c(\mathbf{r}) = \frac{\hbar^2}{2M}\left\{K(\mathbf{r}) - H^2(\mathbf{r})\right\} \tag{2}$$

where $K(\mathbf{r})$ is the Gaussian curvature and $H(\mathbf{r})$ is the mean curvature of the curved surface.[1]

In a recent paper [10], we have studied quantum mechanics in the field of a dispiration (a combined structure of a dislocation and a disclination), and have pointed out that the path integral calculation leads to a result different from that of the Schrödinger equation in the Podolsky form with no curvature term. The purpose of the present Letter is to show that for a particle moving in conical space the path integral calculation is consistent with the Schrödinger equation only when the curvature term is of the Jensen–Koppe form (2). We find that the curvature effect of the conical surface gives rise to an effective potential in the radial path integral and that the effective potential is proportional to the squared mean curvature of the conical surface.

2. Conical space

We consider a two-dimensional curved space with metric,

$$dl^2 = dr^2 + \sigma^2 r^2\, d\theta^2, \tag{3}$$

where $0 \leqslant r$ and $\theta \in [0, 2\pi)$, σ being a real parameter. If $|\sigma| \leqslant 1$, this space may be realized as a conical surface when imbedded in the three-dimensional Euclidean space with

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[1] For more recent and more general discussions, see [6–9].

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$$x = \sigma r \cos\theta,$$
$$y = \sigma r \sin\theta,$$
$$z = \sqrt{1-\sigma^2}\, r. \tag{4}$$

Obviously the limiting case where $\sigma = 1$ corresponds to the two-dimensional flat space with $z = 0$.

The medium around an axial wedge dislocation in solid may be characterized by such a space, for which σ is related to the deficit angle γ via $\sigma = 1 - \gamma/2\pi$. See, e.g., [10,11]. Another example is the space surrounding a massive cosmic string with a linear energy density η, given in a weak field approximation, for which $\sigma = 1 - 4G\eta$ where G is the Newtonian gravitational constant [12]. The conical topology also occurs in $(2+1)$-dimensional Einstein gravity with localized masses [13]. In this connection, the Schrödinger equation in conical space has been further discussed in [14–16].

In the present Letter, however, we study quantum mechanics in the space with metric (3) by path integration. Specifically we carry out path integration for a particle with mass M moving in the conical space with metric (3) under the influence of a two-dimensional central potential $V(r)$. The Lagrangian for the particle is

$$L = \frac{M}{2}\left(\dot r^2 + \sigma^2 r^2 \dot\theta^2\right) - V(r). \tag{5}$$

Although the potential $V(r)$ will appropriately be chosen later for explicit calculation, we assume that it contains a short-ranged repulsive part. In fact, the Hamiltonian describing a free motion in a conical space has a one-parameter family of self-adjoint extensions and requires a careful definition of boundary conditions imposed at $r = 0$. Such boundary conditions are to be determined by the physics near $r = 0$ [15]. We circumvent such a singularity problem at $r = 0$ by assuming an appropriate short-ranged repulsive potential.

3. The propagator

What we wish to calculate by path integration is the propagator or Feynman's kernel for the aforementioned system. Feynman's path integral for the propagator may be given in the time-sliced form [17],

$$K\left(\mathbf{r}'', \mathbf{r}'; \tau\right) = \lim_{N\to\infty} \int \prod_{j=1}^{N-1} d^2\mathbf{r}_j \prod_{j=1}^{N} K(\mathbf{r}_j, \mathbf{r}_{j-1}; \varepsilon) \tag{6}$$

where the time interval $\tau = t'' - t'$ is sliced into N short-time intervals $\varepsilon = \tau/N$. The short time propagator is given by

$$K(\mathbf{r}_j, \mathbf{r}_{j-1}; \varepsilon) = \frac{M\sigma}{2\pi i\hbar\varepsilon} e^{(i/\hbar)S_j} \tag{7}$$

with a short-time action,

$$S_j = \int_{t_{j-1}}^{t_j} L\, dt = \frac{M}{2\varepsilon}\Delta\mathbf{r}_j^2 - V(r_j)\varepsilon. \tag{8}$$

The amplitude of the short-time propagator (7) has been determined by the condition,

$$\lim_{\varepsilon\to 0} K(\mathbf{r}_j, \mathbf{r}_{j-1}; \varepsilon) = \delta(\mathbf{r}_j - \mathbf{r}_{j-1}). \tag{9}$$

Here we intend to carry out path integration for (6) in polar coordinates [18–20] using the relations [10],

$$\prod_{j=1}^{N-1} d^2\mathbf{r}_j = \prod_{j=1}^{N-1} r_j\, dr_j\, d\theta_j,$$
$$\Delta\mathbf{r}_j^2 = \Delta r_j^2 + 2\sigma^2 \hat r_j^2 (1 - \cos\Delta\theta_j) \tag{10}$$

where $\hat r_j = (r_j r_{j-1})^{1/2}$. The angular integration is easily performed by utilizing the generating function for the modified Bessel function,

$$\exp\{z\cos\Delta\theta_j\} = \sum_{m=-\infty}^{\infty} e^{im\Delta\theta_j} I_m(z), \tag{11}$$

valid for any complex number z. As a result of the successive angular integrations, the propagator (6) can be expressed in the form of the partial wave expansion,

$$K\left(\mathbf{r}'', \mathbf{r}'; \tau\right) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\theta''-\theta')} R_m\left(r'', r'; \tau\right). \tag{12}$$

The radial propagator for the m-th partial waves still remains to be path-integrated by

$$R_m\left(r'', r'; \tau\right) = \lim_{N\to\infty}\int \prod_{j=1}^{N-1} r_j\, dr_j \prod_{j=1}^{N} R_m(r_j, r_{j-1}; \varepsilon) \tag{13}$$

with the short-time radial propagator

$$R_m(r_j, r_{j-1}; \varepsilon) = \frac{M}{i\hbar\varepsilon}\exp\left\{\frac{i}{\hbar}\left[\frac{M}{2\varepsilon}\left(r_j^2 + r_{j-1}^2\right) - V(r_j)\varepsilon\right]\right\}\mathbf{I}_m^\sigma(\hat r_j) \tag{14}$$

where

$$\mathbf{I}_m^\sigma(\hat r_j) = \sigma\exp\left\{\frac{iM}{\hbar\varepsilon}\left(\sigma^2 - 1\right)\hat r_j^2\right\} I_m\left(\frac{M\sigma^2}{i\hbar\varepsilon}\hat r_j^2\right). \tag{15}$$

Notice that the parameter σ appears in (14) only though the function (15) and that

$$\mathbf{I}_m^1(\hat r_j) = \lim_{\sigma\to 1}\mathbf{I}_m^\sigma(\hat r_j) = I_m\left(\frac{M}{i\hbar\varepsilon}\hat r_j^2\right). \tag{16}$$

In the limit $\sigma \to 1$, the radial propagator (13) becomes that in flat space. Accordingly, (16) is the modified Bessel function to appear in a flat space path integral. Hence the effect of the deviation from the flat space path integral is all contained in (15). In what follows, using the asymptotic recombination technique, we shall show that the deviation from the flat space gives rise to an effective potential in the radial path integral.

4. The curvature effect

A very useful calculation method for a polar coordinate path integral is the asymptotic recombination technique [21], to which basic is the one-term asymptotic formula of the modified Bessel function, originally employed by Edwards and Gulyaev [18],

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\exp\left\{z - \frac{1}{2z}\left(\nu^2 - \frac{1}{4}\right)\right\}, \tag{17}$$

valid for large $|z|$ with $\mathrm{Re}\, z > 0$. In a polar coordinate path integral, the complex variable z is of the form $z = Mar^2/(i\hbar\varepsilon)$ where a is a positive real constant. In order to justify the use of the asymptotic formula, it is necessary to assume that $M/\hbar$ has a small positive imaginary part $\mathrm{Im}(M/\hbar) > 0$. The same trick has been used in the standard analytic continuation procedure to obtain a well-defined Feynman path integral. Of course, in this case, $|z|$ is large when ε

is small. It is clear that the asymptotic formula (17) is valid insofar as it is used for a short-time Feynman path integral.

Substituting (17) with $z = M\sigma^2\hat{r}_j^2/(i\hbar\varepsilon)$ into (15), combining the exponential factor of (15) with (17), and rearranging terms, we find the result,

$$\mathbf{I}_m^\sigma(\hat{r}_j) \sim \exp\left\{-\frac{i}{\hbar}V_{eff}(\hat{r}_j)\varepsilon\right\} I_{m/\sigma}\left(\frac{M}{i\hbar\varepsilon}\hat{r}_j^2\right) \tag{18}$$

where

$$V_{eff}(\hat{r}_j) = -\frac{\hbar^2}{2M}\frac{1-\sigma^2}{4\sigma^2\hat{r}_j^2}. \tag{19}$$

Correspondingly, the short-time radial propagator (14) can be written as

$$R_m(r_j, r_{j-1}; \tau) = \frac{M}{i\hbar\varepsilon}\exp\left\{\frac{i}{\hbar}\left[\frac{M}{2\varepsilon}\left(r_j^2 + r_{j-1}^2\right) - V(r_j)\varepsilon - V_{eff}(r_j)\varepsilon\right]\right\}\mathbf{I}_{m/\sigma}^1(\hat{r}_j). \tag{20}$$

This means that the radial path integral in a conical space can be understood as a flat space path integral modified with the effective potential (19). Furthermore, the integral angular momentum m of each partial wave is replaced by a non-integral angular momentum m/σ. Note that m/σ is not assumed to be an integer or a fractional number but is yet countable. The angular summation (12) for the full propagator is still over $m \in \mathbf{Z}$.

The appearance of the effective potential $V_{eff}(r)$ in (20) as an addition to the assumed external potential $V(r)$ is remarkable. What is more remarkable is however that (19) is identifiable with the effect of the curvature of the conical surface imbedded in Euclidean space. The mean curvature and the Gaussian curvature of the cone are given by

$$H(r) = \frac{\sqrt{1-\sigma^2}}{2\sigma r} \quad \text{and} \quad K(r) = 2\pi\frac{1-\sigma}{\sigma}\delta^{(2)}(x, y), \tag{21}$$

respectively, where $\delta^{(2)}(x, y)$ represents the two-dimensional δ-function (see [10] for detail). Apparently the effective potential (19) can be expressed as

$$V_{eff}(r) = -\frac{\hbar^2}{2M}H^2(r). \tag{22}$$

In the presence of a repulsive potential which regularizes the singularity at $r = 0$, i.e. propagator and wave function vanish at $r = 0$, the Gaussian curvature has no effect and we can put (22) into the form

$$V_{eff}(r) = \frac{\hbar^2}{2M}\left\{K(r) - H^2(r)\right\}. \tag{23}$$

The last expression is indeed identical in form with the curvature term (2) in the Schrödinger equation. This means that the path integral calculation in conical space is consistent only with the Schrödinger equation modified by the curvature term of the form of (2).

5. The energy spectrum and the wave functions

In the above, we have expressed the short-time radial propagator in the form (20) in order to emphasize the emergence of the effective potential which is identifiable with the mean curvature of the conical surface. Since the effective potential as is given by (19) is of the inverse square form $\sim 1/r^2$, it can be absorbed via the asymptotic recombination into the index of the modified Bessel function, so that the function (18) can be given by

$$\mathbf{I}_m^\sigma(\hat{r}_j) \sim I_{\mu(m)}\left(\frac{M}{i\hbar\varepsilon}\hat{r}_j^2\right) \tag{24}$$

with

$$\mu(m) = \frac{1}{2\sigma}\sqrt{4m^2 + \sigma^2 - 1}. \tag{25}$$

Thus the short-time radial propagator may also be written as

$$R_m(r_j, r_{j-1}; \varepsilon) = \frac{M}{i\hbar\varepsilon}\exp\left\{\frac{i}{\hbar}\left[\frac{M}{2\varepsilon}\left(r_j^2 + r_{j-1}^2\right) - V(r_j)\varepsilon\right]\right\} \times I_\mu\left(\frac{M}{i\hbar\varepsilon}\hat{r}_j^2\right). \tag{26}$$

Here the cone topology is completely encoded in the index of the modified Bessel function. In other words, the short-time radial propagator (26) is formally identical with that of flat space with angular momentum $m \in \mathbf{Z}$ replaced by an effective angular momentum $\mu(m) \in \mathbf{R}$. It is important to notice that whenever the centrally symmetric problem in flat space is soluble by path integration the corresponding problem on the conical space can also be solved by path integration.

To carry out the radial path integration, we have to specify the external potential $V(r)$. For convenience to our discussion, we consider a combination of the harmonic oscillator potential and a repulsive inverse-square potential,

$$V(r) = \frac{1}{2}M\omega^2 r^2 + \frac{\kappa\hbar^2}{8\sigma^2 M r^2}, \tag{27}$$

where $\kappa > 1 - \sigma^2 > 0$. This represents long-range attraction and short-range repulsion. The repulsive potential with the chosen κ removes the singularity at $r = 0$. In fact the finite-time radial path integral for the potential (27) in flat space has been explicitly evaluated [19,22–24], the result being

$$R_m(r'', r'; \tau) = \frac{M\omega}{2\pi i\hbar\sin\omega\tau}\exp\left\{\frac{iM\omega}{2\hbar}\left(r'^2 + r''^2\right)\cot\omega\tau\right\} \times I_{\nu(m,1)}\left(\frac{M\omega}{i\hbar\sin\omega\tau}r'r''\right) \tag{28}$$

with

$$\nu(m, 1) = \frac{1}{2}\sqrt{4m^2 + \kappa}. \tag{29}$$

As is mentioned above, the finite-time radial propagator in the conical space can be immediately obtained from (28) by simply replacing $\nu(m, 1)$ by

$$\nu(m, \sigma) = \frac{1}{2\sigma}\sqrt{4\nu^2(m, 1) + \sigma^2 - 1} = \frac{1}{2\sigma}\sqrt{4m^2 + \kappa + \sigma^2 - 1}. \tag{30}$$

The desired result is therefore identical with (28) if $\nu(m, \sigma)$ of (30) is in the place of $\nu(m, 1)$.

With the help of (12) and the Hille–Hardy formula [25], the full propagator can be expressed in the spectral representation,

$$K(\mathbf{r}'', \mathbf{r}'; \tau) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} e^{-(i\tau/\hbar)E_{nm}}\Psi_{nm}^*(r', \theta')\Psi_{nm}(r'', \theta'') \tag{31}$$

with the energy spectrum and the energy eigenfunctions given, respectively, by

$$E_{nm} = \hbar\omega\left(2n + 1 + \frac{1}{2\sigma}\sqrt{4m^2 + \kappa + \sigma^2 - 1}\right) \tag{32}$$

and

$$\Psi_{nm}(r,\theta) = N_{nm} e^{im\theta} r^{\nu(m,\sigma)} e^{-(M\omega/2\hbar)r^2} \times {}_1F_1\big(-n, \nu(m,\sigma)+1; (M\omega/\hbar)r^2\big) \qquad (33)$$

where

$$N_{nm} = \frac{1}{\Gamma(\nu(m,\sigma)+1)} \sqrt{\frac{\Gamma(n+\nu(m,\sigma)+1)}{\pi n!}} \left(\frac{M\omega}{\hbar}\right)^{(\nu+1)/2}. \qquad (34)$$

As we have assumed $\kappa \geqslant 1-\sigma^2$ for making the system singularity-free, the energy eigenvalues (32) are assured to be real. Since $\nu(m,\sigma)$ in (30) also remains real, the wave functions (33) vanish at $r=0$ for $\nu(m,\sigma) \neq 0$.

6. Conclusion

In the present Letter path integration has been carried out explicitly for a particle in the field of a topological defect characterized by the conical metric (3). By the asymptotic recombination technique applied to the short-time path integral, we have shown that the quantum mechanics in a conical space is essentially identical to that in flat space with rescaled angular momentum, and that the path integration calculation naturally leads to an effective potential which is proportional to the squared mean curvature of the conical space imbedded in Euclidean space. We have seen that our path integral calculation is consistent with the Schrödinger equation modified by the Gaussian and mean curvatures as in [4].

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